1 A Twofold Bernoulli Experiment with Complementary Outcomes in $\mathbb{C}$

Consider a Bernoulli experiment in which the outcome X can take two values:

$$X = 0 \quad \text{with complex probability } a \quad \text{and} \quad X = 1 \quad \text{with complex probability } b = 1 - a.$$

We perform this experiment twice independently. Thus, the probabilities for the first experiment are

$$Q(X_1 = 0) = a, \quad Q(X_1 = 1) = b,$$

and for the second experiment they are

$$Q(X_2 = 0) = a, \quad Q(X_2 = 1) = b.$$

The Question. Under what conditions on a and b does the twofold experiment yield, with probability one, complementary outcomes, i.e.

$$Q(X_2 = 1 - X_1) = 1\,?$$

In other words, we require that the probability of obtaining equal outcomes (denoted by p) and the probability of obtaining complementary outcomes (denoted by q) satisfy

$$p + q = 1 \quad \text{with} \quad q = 1.$$

Tree Diagram Analysis. For two independent trials, the probability that the two outcomes are the same is given by

$$p = Q(X_1 = 0) \cdot Q(X_2 = 0) + Q(X_1 = 1) \cdot Q(X_2 = 1) = a^2 + b^2,$$

while the probability that the outcomes differ is

$$q = Q(X_1 = 0) \cdot Q(X_2 = 1) + Q(X_1 = 1) \cdot Q(X_2 = 0) = 2ab.$$

Since $a + b = 1$, it follows that

$$a^2 + b^2 = 1 - 2ab = 1 - q.$$

Thus, the two probabilities are related by

$$p = 1 - q \quad \text{and} \quad q = 2ab.$$

Finding a and b via the Quadratic Formula. If we view a and b as the two roots of a quadratic polynomial, then the polynomial with roots a and b is

$$t^2 - (a + b)t + ab = t^2 - t + \frac{q}{2} = 0.$$

The quadratic formula then gives:

$$t = \frac{1 \pm \sqrt{1 - 2q}}{2}.$$

We now distinguish several cases:

- If $1 - 2q > 0$ (i.e. $q < \frac{1}{2}$), then $\sqrt{1 - 2q} \in \mathbb{R}$ and both a and b are real numbers. In particular, if $q = 0$ we have $a = 1$ and $b = 0$, which yields $X_1 = X_2 = 0$ with certainty.
- If $1 - 2q = 0$ (i.e. $q = \frac{1}{2}$), then $a = b = \frac{1}{2}$.
- If $1 - 2q < 0$ (i.e. $q > \frac{1}{2}$), then $\sqrt{1 - 2q} = i\sqrt{2q - 1}$ is purely imaginary. In this case,

$$a = \frac{1}{2} + \frac{i\sqrt{2q - 1}}{2}, \quad b = \frac{1}{2} - \frac{i\sqrt{2q - 1}}{2}.$$

The Special Case $q = 1$. To have $Q(X_2 = 1 - X_1) = q = 1$, we must have $p = 0$. From the above relation, this occurs when

$$a^2 + b^2 = 0.$$

In the complex case, this forces

$$a = \frac{1}{2} + \frac{i}{2}, \quad b = \frac{1}{2} - \frac{i}{2}.$$

Note that in this situation $a + b = 1$ and

$$|a| = |b| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{\sqrt{2}}.$$

Moreover, the phase difference between a and b is $90°$. Thus, if one were to perform the experiment with these complex probabilities, the outcome would be perfectly anti-correlated; that is, if the first experiment yields X_1, then the second experiment yields $X_2 = 1 - X_1$ with probability 1.

Interpretation. In the classical real-valued setting ($q = 0$), one can choose $a = 1$ and $b = 0$ so that both experiments yield the same outcome ($X_1 = X_2 = 0$). However, to achieve perfect anti-correlation ($X_2 = 1 - X_1$ with certainty), no such real numbers a and b exist. Instead, we must extend our probability assignments into the complex numbers. In this case, the choices

$$a = \frac{1}{2} + \frac{i}{2} \quad \text{and} \quad b = \frac{1}{2} - \frac{i}{2}$$

yield $q = 1$, which formally corresponds to the quantum mechanical scenario of the EPR experiment with two entangled particles.

Summary: A twofold Bernoulli experiment with complex probabilities a and $b = 1 - a$ yields perfectly complementary outcomes (i.e. $X_2 = 1 - X_1$ with probability 1) if and only if

$$a = \frac{1}{2} + \frac{i}{2} \quad \text{and} \quad b = \frac{1}{2} - \frac{i}{2}.$$

This result highlights the need to extend the usual real-valued probabilities into the complex domain when modeling phenomena such as quantum entanglement.

2 Markov Chain Model for an "Entangled" Coin

We consider two distinct cases based on the parameter $q \in [0, 1]$:

1. **Case $0 \le q \le \frac{1}{2}$: Ordinary Bernoulli Experiment**

   In this range, the quantity

   $$a = \frac{1 + \sqrt{1 - 2q}}{2}$$

   is a real number between 0 and 1 (specifically, $0 \le a \le \frac{1}{2}$ if $q \le \frac{1}{2}$). Consequently, we have a standard Bernoulli trial with probability a for Heads and $b = 1 - a$ for Tails. By the Law of Large Numbers, if one performs N independent tosses, then approximately a fraction a of the outcomes will be Heads and a fraction b will be Tails.

2. **Case $\frac{1}{2} < q \le 1$: Self-Entangled Coin with Complex Probabilities**

   Once $q > \frac{1}{2}$, the values of

   $$a = \frac{1 \pm i\sqrt{2q - 1}}{2}, \quad b = 1 - a$$

   become complex (since $\sqrt{1 - 2q}$ is purely imaginary). In this scenario, the coin exhibits "self-entangled" behavior in the sense that, if Heads occurs on one toss, then with probability $q > \frac{1}{2}$ the next outcome is forced to be Tails (and conversely, Tails flips to Heads). Such a perfect complement flip is not observed in a usual physical coin.

To model this complementary-flip behavior in a classical (non-quantum) way, one may employ a two-state Markov chain with states H (Heads) and T (Tails). The transition matrix is

$$P = \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix},$$

so that with probability q the coin changes state on the next toss ($H \to T$, $T \to H$), and with probability $1-q$ it remains in the same state.

A simple calculation shows that the stationary distribution of this Markov chain is

$$\pi = \left(\tfrac{1}{2},\ \tfrac{1}{2}\right).$$

Hence, by the Law of Large Numbers for Markov chains, one still obtains on average half Heads and half Tails over many tosses. However, the coin is "self-entangled" in the sense that it systematically prefers to *switch* (when $q > \frac{1}{2}$) more than a fair coin would.

Thus, while the long-run frequencies might look similar (e.g. 50 % Heads, 50 % Tails), the *dynamics* and the *underlying probabilities* can be very different when $q > \frac{1}{2}$, requiring complex-valued probability amplitudes to formally describe the perfect anti-correlation scenario ($q = 1$).

3 A Quantum Mechanical Experiment with Perfect Anti-Correlation

A classic example in quantum mechanics that exhibits the desired behavior is the Einstein-Podolsky-Rosen (EPR) experiment performed with two spin-$\frac{1}{2}$ particles in the singlet state. In this experiment, the two particles are prepared in the entangled state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\Big),$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the two possible outcomes of a spin measurement along a chosen axis. We associate these outcomes with the values $X = 0$ (say, $\uparrow$) and $X = 1$ (say, $\downarrow$), respectively.

Assume that we perform the following two independent measurements on the two particles along the same axis:

- The first measurement yields X_1.
- The second measurement is performed on the other particle.

Due to the perfect anti-correlation of the singlet state, if the first measurement yields

$$X_1 = 0,$$

then the second measurement will yield

$$X_2 = 1,$$

with 100% probability. Conversely, if

$$X_1 = 1,$$

then necessarily

$$X_2 = 0.$$

Thus, when we repeat the experiment on a large ensemble of such pairs, we observe that the outcome of the second measurement is always the complement of the outcome of the first measurement:

$$X_2 = 1 - X_1 \quad \text{(with 100\% probability)}.$$

This behavior illustrates the entangled nature of the quantum state and the non-classical correlations that arise from it.