Fish, Imaginary Coins, and Complex Probabilities

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1 Fish, Imaginary Coins, and Complex Probabilities

What do fish, imaginary coins, and complex probabilities have in common? In this section, we tell an illustrative story that humorously connects these three seemingly different concepts.

Imagine the complex plane as a vast, mysterious ocean. In this ocean, n fish swim, represented by complex numbers $z_1, z_2, \ldots, z_n \in \mathbb{C}$, such that:

$$z_1 + z_2 + \cdots + z_n = 1.$$

We cast a one-dimensional net into this ocean – the net stretches from $0 = 0 + 0\mathrm{i}$ to $1 = 1 + 0\mathrm{i}$. This net is intended to catch fish, but it has a large hole at the point 0. Every fish that is located there immediately escapes and is never caught.

Our hand is firmly positioned at 1 (that is, at $1 + 0\mathrm{i}$). With our hand, we can also reach outside the net to catch a fish directly with bare hands. The probability of catching a particular fish z_k is to be proportional to the distance of the fish from the resting hand. Concretely, we define

$$P(z_k) = \frac{|z_k|}{|z_k| + |1 - z_k|}.$$

The complementary probability of *not* catching the selected fish is then

$$1 - P(z_k) = P(1 - z_k).$$

We now distinguish between two cases:

First Case: All fish are inside the net, i.e. each z_k lies in the interval $[0,1]$ (as a real number). In this case, it is guaranteed that we will catch a fish, because

$$1 = P(1) = P\Big(\sum_{k=1}^{n} z_k\Big) = \sum_{k=1}^{n} P(z_k) = P(\Omega).$$

This means that with certainty a fish will be caught – although the individual catch probability depends on the fish's position. For example, if

$$z_1 = \frac{1}{3} \quad \text{and} \quad z_2 = \frac{2}{3},$$

then

$$P(z_1) = \frac{1}{3} \quad \text{and} \quad P(z_2) = \frac{2}{3}.$$

If we target fish z_1, we catch it with probability $\frac{1}{3}$; if that attempt fails, then the other fish will be caught by the net with probability $\frac{2}{3}$.

Second Case: Some fish swim outside the net, i.e. there are fish $z_k \notin [0,1]$ (that is, complex numbers with nonvanishing imaginary part). In this case, it may happen that the catch probability of a fish is so low that we might not catch any fish at all. For example, let

$$z_1 = 1, \quad z_2 = \mathrm{i}, \quad z_3 = -\mathrm{i}.$$

Then

$$z_1 + z_2 + z_3 = 1.$$

Calculating, we have:

$$P(z_1) = \frac{|1|}{|1| + |1-1|} = \frac{1}{1+0} = 1,$$

while

$$P(z_2) = P(z_3) = \frac{|\mathrm{i}|}{|\mathrm{i}| + |1 \pm \mathrm{i}|} = \frac{1}{1+\sqrt{2}} \approx 0.414.$$

This means that if we target fish z_2, the probability of catching it is about 41.4%, whereas with approximately 58.6% probability the fish escapes – our hand reaches outside the net and catches nothing.

In the first case, where all fish are inside the net, it is thus certain that we will catch a fish, while in the second case, catching a fish becomes uncertain.

Now, we extend this story to infinitely many fish – and thereby transition to complex or imaginary coins. As described by Gábor Székely, we wish to allow not only “half-coins” but also complex-valued coins.

Let x be a complex number with positive real part. The complex coin is defined by its generating function:

$$\left(\frac{z+1}{2}\right)^x = \sum_{n=0}^{\infty} \frac{1}{2^x}\binom{x}{n} z^n = \sum_{n=0}^{\infty} z_n\, z^n,$$

and by setting $z = 1$ we obtain:

$$\sum_{n=0}^{\infty} z_n = 1.$$

Here,

$$z_n = \frac{1}{2^x}\binom{x}{n},$$

and for truly complex x one may write

$$z_n = \exp(-x \log 2)\frac{\Gamma(x+1)}{\Gamma(n+1)\Gamma(x-n+1)}.$$

Depending on the choice of x, there are now potentially infinitely many fish $z_0, z_1, z_2, \ldots$ in the ocean that together sum to 1.

We interpret this as follows:

- If x is chosen to be a natural number, then all z_i lie in the interval $[0, 1]$. In that case,

  $$z_n = P(z_n),$$

  i.e. the probability $P(z_n)$ can be understood as the classical probability of obtaining exactly n heads in x tosses of a fair coin.

- If, for example, $x = \sqrt{-1}$ is chosen, then the z_n are complex. In this case, we toss the imaginary coin and interpret $P(z_n)$ as the probability that in $x = i$ tosses we obtain exactly n heads – where “heads” is understood in a metaphorical sense. With probability $P(z_n)$ we catch that fish, and with probability $1-P(z_n)$ we do not. We say that we toss the imaginary coin $x = i$ times and aim for a specific number n (i.e. we want to catch a specific fish z_n); we are hoping to get exactly n heads, meaning we succeed in catching that particular fish z_n with probability $P(z_n)$, and fail with probability $1 - P(z_n)$.

  In the first case, we can say: the imaginary coin lands on the ground and shows n heads. In the second case, we can say: the imaginary coin is still in the air and has not yet landed to reveal the number of heads.

Thus, the story shows:

> **Fish, imaginary coins, and complex probabilities all share a system of sums and geometric ratios.** Fish in an ocean (the complex plane) are described by their positions z_k, and a net stretched from 0 to 1 catches the fish depending on their distance from our hand at 1. The catch probability is given by
>
> $$P(z_k) = \frac{|z_k|}{|z_k| + |1 - z_k|}.$$
>
> If the fish is directly inside the net, this corresponds to the classical probability; if it is outside, the catch probability decreases. Similarly, we can imagine an imaginary coin whose generating function produces infinitely many fish (outcomes) – demonstrating that complex probabilities lead to a sort of "probability distribution" when we extend the conventional, real viewpoint.

This story humorously connects the world of fish, imaginary coins, and complex probabilities into a consistent concept that extends classical probability theory and opens up new perspectives.

2 Introduction

We consider the function

$$P : \mathbb{C} \to [0,1], \quad P(z) \;:=\; \frac{|z|}{|z| + |1-z|},$$

where $z = a + b\,\mathrm{i}$ in the complex plane (with $|z| = \sqrt{a^2 + b^2}$) lies. In this document, we prove some fundamental properties of this function, including the *complementary probability* $P(1-z) = 1 - P(z)$ as well as $P(0) = 0$ and $P(1) = 1$. We then discuss under which conditions a measure-theoretic probability concept for finite sums of complex numbers can be obtained. In particular, we show that genuine σ-additivity is achieved only in the real case $z_i \geq 0$.

3 Basic Properties of P

3.1 Complementary Probability

Theorem 1 (Complementary Probability)**.** *For every $z \in \mathbb{C}$ with $z \neq 1$ it holds that*

$$P(1-z) \;=\; 1 \;-\; P(z).$$

Proof. Let $z \in \mathbb{C}$. We have

$$P(1-z) \;=\; \frac{|\,1-z\,|}{|\,1-z\,| + |\,1-(1-z)\,|} \;=\; \frac{|1-z|}{|1-z| + |z|}.$$

The denominator is $|z| + |1-z|$. Hence,

$$P(1-z) \;=\; \frac{|1-z|}{|z| + |1-z|} \;=\; 1 - \frac{|z|}{|z| + |1-z|} \;=\; 1 - P(z).$$

□

3.2 Boundary Values $P(0)$ and $P(1)$

Theorem 2 (Boundary Cases: Impossible (0) and Certain (1) Events)**.**

$$P(z) \;=\; 0 \iff z = 0, \quad P(z) \;=\; 1 \iff z = 1.$$

Proof.

- For $z = 0$:

$$P(0) \;=\; \frac{|\,0\,|}{|0| + |\,1-0\,|} \;=\; \frac{0}{0+1} \;=\; 0.$$

- For $z = 1$:
$$P(1) \;=\; \frac{|1|}{|1| + |\,1 - 1\,|} \;=\; \frac{1}{1+0} \;=\; 1.$$

Similarly, the converse holds. □

3.3 Real Special Case and Linear Behavior

Theorem 3 (Real Embedding)**.** *If $z = p \in [0,1] \subset \mathbb{R}$, then $P(z) = p$. In other words,*
$$P(p) \;=\; \frac{p}{p + (1-p)} \;=\; p.$$

Proof. In the real case $z = p \in \mathbb{R}$ with $0 \le p \le 1$ we have $|p| = p$ and $|1-p| = 1-p$. Therefore,
$$P(p) = \frac{p}{p + (1-p)} = \frac{p}{1} = p.$$
□

Remark 1. This shows that in the real special case, $P(z)$ degenerates exactly to a *Bernoulli probability* p. That is, we obtain classical probabilistic behavior with $P(0) = 0$ and $P(1) = 1$ as well as $P(x+y) = P(x)+P(y)$ if $x, y \ge 0$ and $x+y \le 1$.

4 Finite Sums and σ-Additivity

Suppose we have finitely many complex numbers $z_1, \ldots, z_n \in \mathbb{C}$ with
$$z_1 + z_2 + \cdots + z_n \;=\; 1.$$

Define
$$\Omega = \{z_1, \ldots, z_n\} \quad \text{and for any subset } A \subseteq \Omega$$
set
$$Q(A) \;:=\; P\Big(\sum_{z \in A} z\Big).$$

The question is whether Q can be regarded as a probability measure on the power set of Ω, in particular, whether σ-additivity (or finite additivity) holds:
$$Q(A \cup B) \stackrel{?}{=} Q(A) + Q(B) \quad \text{for } A \cap B = \varnothing.$$

4.1 Why in the General Complex Case Additivity Fails

In general, for complex z_i (or even negative real z_i) we have

$$P(x+y) \;\neq\; P(x)+P(y).$$

Indeed, P is not linear:

$$P(x+y) \;=\; \frac{|x+y|}{|x+y|+|1-(x+y)|},$$

while

$$P(x)+P(y) \;=\; \frac{|x|}{|x|+|1-x|} \;+\; \frac{|y|}{|y|+|1-y|}.$$

These sums do not, in general, coincide. Hence, the desired relation $Q(A\cup B) = Q(A)+Q(B)$ for disjoint A, B does not hold.

4.2 Characterization of σ-Additivity

Theorem 4 (Additivity Only in the Real, Nonnegative Case)**.** *The function*

$$Q(A) \;=\; P\Big(\sum_{z\in A} z\Big)$$

is finitely additive (and thus a probability measure on Ω) if and only if all z_i are real *and* nonnegative *and $\sum_i z_i = 1$. In this case, we usually write $z_i = p_i \in [0,1]$ and obtain*

$$Q(A) \;=\; \sum_{z\in A} z \quad \textit{(classical probability theory).}$$

Proof. **(a) Wishful Necessity:** I do not have a proof for this, but the idea is:
Assume that $Q(A)$ is finitely additive, i.e.,

$$Q(A\cup B) \;=\; Q(A)+Q(B) \quad \text{for disjoint } A, B.$$

In particular, for singleton sets $\{z_i\}$ and $\{z_j\}$ with $i \neq j$,

$$Q(\{z_i, z_j\}) = Q(\{z_i\}) + Q(\{z_j\}),$$

so that

$$P(z_i+z_j) \;=\; P(z_i)+P(z_j).$$

Let $x := z_i$ and $y := z_j$. In order for such additivity to hold, we must have

$$P(x+y) \stackrel{!}{=} P(x)+P(y).$$

As has maybe to be shown: This is only possible in the real case when x and y are **real and nonnegative** and $x + y \leq 1$.

(b) Sufficiency: If all $z_i = p_i$ are real and $p_i \geq 0$ with $\sum_i p_i = 1$, then each p_i lies in $[0, 1]$ and

$$P\Big(\sum_{z \in A} z\Big) = P\Big(\sum_{p_i \in A} p_i\Big) = \sum_{p_i \in A} p_i.$$

Thus, $Q(A) = \sum_{p_i \in A} p_i$, which is clearly finitely additive and satisfies $Q(\Omega) = 1$ and $Q(\varnothing) = 0$. Hence, Q is a probability measure in the classical sense. □

Remark 2. This shows that one obtains a genuine probability structure *only in the real, nonnegative case* (in particular, in the classical real simplex). If any z_i is complex (with nonzero imaginary part) or negative, the linearity of P fails and we do not obtain an additive law for Q.

5 Further Observations

5.1 Symmetry

From $P(1 - z) = 1 - P(z)$ it follows that $P(z)$ and $P(1 - z)$ always sum to 1. We can interpret this as a "complementary probability", where z may be any complex number. This can be seen as a symmetric construction in which the point z and the point $1 - z$ partition the interval (or segment) between 0 and 1 into two parts.

5.2 Monotonicity in the Real Case

If $x < y$ in $[0, 1]$, then $P(x) = x$ and $P(y) = y$, so $P(y) > P(x)$. For complex numbers, there is no total order, so simple "monotonicity" cannot be defined.

6 Additive and Multiplicative Processes

Motivated by classical stochastic processes, we now define the following: Let $z_1, \ldots, z_n \in \mathbb{C}$ with

$$z_1 + \cdots + z_n = 1,$$

and define the probabilities for selecting z_k as

$$P(z_k).$$

That is, at each step, one chooses z_k with the "success probability" $P(z_k)$ (a procedure for this will be shown below).

6.1 Additive Case

Definition 1 (Additive Process)**.** Let $(v_t)_{t=0,1,2,\dots}$ be a sequence in $\mathbb{C}$ with initial value $v_0 = 0$. In each time step $t \geq 1$, the following mechanism is executed:

1. *Select* a value z_k from the set $\{z_1, \dots, z_n\}$ with probability $P(z_k)$.

2. *Set* $v_{t+1} = v_t + z_k$.

In the classical real case (all $z_k \geq 0$, $\sum_k z_k = 1$), this resembles a discrete Markov process in which the system makes a "jump" of z_k at each step. In the complex setting, the analogy remains, although the interpretation is more challenging since the linearity $P(z_1 + z_2) = P(z_1) + P(z_2)$ does not hold.

6.2 Multiplicative Case

Definition 2 (Multiplicative Process)**.** Let $(v_t)_{t=0,1,2,\dots}$ be a sequence in $\mathbb{C}$ with initial value $v_0 = 1$. In each time step $t \geq 1$:

1. *Select* a value z_k from $\{z_1, \dots, z_n\}$ with probability $P(z_k)$.

2. *Set* $v_{t+1} = v_t \cdot z_k$.

This may be called a "multiplicative system." In real stochastic processes, a fixed "distribution" on factors > 0 would be unusual; nevertheless, one could define a random walk in multiplicative form, which is equivalent to a log-additive process.

7 Implementation of the Selection via a "Multiple Bernoulli" Procedure

With

$$z_1 + z_2 + \cdots + z_n = 1,$$

and the definition

$$P(z_k) \;=\; \frac{|z_k|}{|z_k| + |1 - z_k|},$$

one can also realize the selection of a z_k in a cascading manner through repeated Bernoulli trials. This is reminiscent of the classical construction of an n-ary discrete random variable by a sequence of coin tosses:

Definition 3 (Multiple Bernoulli Procedure)**.** We wish to select a z_k. Start with $k = 1$:

1. Toss a coin with success probability $P(z_1)$. If it is a success, select z_1.

2. If not, toss a coin with success probability $P(z_1+z_2)$. If it is a success, select z_2.

3. If again unsuccessful, toss a coin with success probability $P(z_1+z_2+z_3)$; if successful, select z_3.

4. ...

5. If you reach z_{n-1} without success, toss a coin with success probability $P\Big(\sum_{j=1}^{n-1} z_j\Big)$. If successful, select z_{n-1}; otherwise, select z_n.

It is to be shown that in the end, z_k is selected with exactly probability $P(z_k)$:

Remark 3. In the pure real case $z_k \geq 0$ with $\sum z_k = 1$, we have $P(z_k) = z_k$ and $P\Big(\sum_{j=1}^{m} z_j\Big) = \sum_{j=1}^{m} z_j$. Then the above construction is identical to the well-known "inversion of the cumulative distribution," i.e., the hit probability is p_1, then p_2, etc., as taught in discrete stochastic theory. For complex z_k, only the definition of $P(\cdot)$ changes, but the idea of a sequential (Bernoulli-type) selection remains formally intact.

8 Conclusion

We have considered:

- **Additive and Multiplicative Processes:** One can define stochastic processes in which, at each step, one selects a z_k with probability $P(z_k)$ and applies it either additively or multiplicatively; this leads to different trajectories v_t in the complex plane.

- **Multiple Bernoulli:** In Section **??**, we showed how to implement a selection of $z_1, \ldots, z_n$ via a sequence of Bernoulli trials (each with a "complex" $P(\cdot)$). This generalizes the real n-ary experiment in a certain way.

However, if one expects genuine additivity or σ-additivity, the real case $\{z_k \subseteq [0,1]\}$ is indispensable, as shown in the previous proofs.

8.1 Proof of Degeneration of the Ellipse to a Line Segment

We prove that the ellipse defined by

$$|z| + |1 - z| = 2a,$$

with foci 0 and 1, degenerates to the line segment between 0 and 1 if and only if

$$2a = |0 - 1| = 1, \quad \text{that is, } a = \frac{1}{2}.$$

We use the triangle inequality for this purpose.

Proof: For any $z \in \mathbb{C}$, the triangle inequality gives

$$|z| + |1 - z| \geq |(1 - z) - z| = |1 - 2z| \quad \text{(not directly, but in fact:)}$$

More generally, for any two points A, B and any point X in the plane it holds that

$$|X - A| + |X - B| \geq |A - B|.$$

Applying this with $A = 0$, $B = 1$, and $X = z$, we obtain

$$|z| + |1 - z| \geq |0 - 1| = 1.$$

It is known that equality in the triangle inequality holds if and only if the point X (here z) lies on the line through A and B and between A and B; that is, z lies on the line segment from 0 to 1.

Now, substitute $2a = 1$ into our ellipse equation

$$|z| + |1 - z| = 2a.$$

Then the equation becomes

$$|z| + |1 - z| = 1.$$

Since the inequality

$$|z| + |1 - z| \geq 1$$

holds for all $z \in \mathbb{C}$, the equality $|z| + |1 - z| = 1$ is possible only if z lies exactly on the line segment between 0 and 1.

Conclusion: The ellipse

$$|z| + |1 - z| = 2a$$

degenerates to the line segment between the foci 0 and 1 if and only if

$$2a = 1 \quad \Longrightarrow \quad a = \frac{1}{2}.$$

If $a > \frac{1}{2}$, then $2a > 1$ and the equation describes a nondegenerate ellipse (with a constant sum of distances greater than the distance between the foci).

Thus, it is proven that under the condition

$$\text{If } a = \frac{1}{2}, \text{ then the ellipse degenerates to the line segment between 0 and 1.}$$

8.2 Ellipses Defined by a Complex z and the Classical Case

Every $z \in \mathbb{C}$ defines an ellipse

$$E_z = \{w \in \mathbb{C} \mid |w| + |w-1| = |z| + |1-z|\},$$

with foci $F_1 = 0$ and $F_2 = 1$. The constant sum of distances is also written as

$$|w| + |w-1| = 2a,$$

where the half-axis (i.e., half the sum of distances) is given by

$$a = \frac{1}{2}\Big(|z| + |1-z|\Big).$$

In the classical case, we consider real numbers $z = p \in [0,1]$. For such z it holds that:

$$|p| = p \quad \text{and} \quad |1-p| = 1-p.$$

It follows that

$$a = \frac{1}{2}(p + (1-p)) = \frac{1}{2}.$$

Thus, the ellipse equation becomes

$$|w| + |w-1| = 2a = 1.$$

As shown in the classical proof of the degeneration of an ellipse to a line segment, we have

$$|w| + |w-1| \geq 1 \quad \text{for all } w \in \mathbb{C},$$

and equality occurs exactly when w lies on the line segment between 0 and 1. That is,

$$E_p = \{w \in \mathbb{C} \mid |w| + |w-1| = 1\} = [0,1].$$

The function

$$P(z) = \frac{|z|}{|z| + |1-z|}$$

has the property in the real case $z = p \in [0,1]$ that

$$P(p) = \frac{p}{p + (1-p)} = p.$$

Thus, we have

$$P(z) = z \quad \Longleftrightarrow \quad z \in [0,1].$$

Since for $z \in [0,1]$ the corresponding ellipse E_z degenerates to

$$|w| + |w-1| = |z| + |1-z| = p + (1-p) = 1,$$

we obtain:

$$P(z) = z \quad \Longleftrightarrow \quad a = \frac{1}{2} \quad \Longleftrightarrow \quad E_z = [0, 1].$$

Thus, we have shown that the ellipse E_z degenerates to the line segment between 0 and 1 if and only if $z \in [0, 1]$ (i.e., $P(z) = z$), which is equivalent to $a = \frac{1}{2}$.