Urn Model with Complex Probabilities

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Abstract

We introduce an urn model in which complex-valued probabilities arise naturally. Initially, there are k balls labeled "Heads" and z balls labeled "Tails," making a total of $n = k + z$. One draws from the urn without replacement until the urn is empty; once it empties, it is refilled with the same configuration of balls and the drawing process continues. Surprisingly, after every block of n draws, one obtains exactly k Heads-balls and z Tails-balls with probability 1. This phenomenon cannot be reproduced by any classical Bernoulli coin with a real-valued bias, but it can be interpreted via a coin whose "Heads" probability is complex. In particular, such a coin faithfully simulates the urn's behavior if and only if the binomial expression

$$\binom{n}{k} p^k (1-p)^{n-k} - 1 \;=\; 0$$

is satisfied by a (generally complex) number p. We discuss how one can construct an equivalent model using dependent Bernoulli variables, effectively mirroring the algebra of complex numbers with pairs of real random variables.

1 Urn Model with Complex Probabilities

Consider the following urn model: we start with k "Heads" balls and z "Tails" balls in an urn, so the total is $n = k + z$. We draw from the urn without replacement until it becomes empty. If the urn is ever empty, we immediately refill it with the original k Heads-balls and z Tails-balls and continue.

Key Observation. After each block of n draws (counting from the beginning), the probability of having drawn exactly k Heads-balls and z Tails-balls in that block is 100%. In more familiar Bernoulli-type experiments, we would never expect such perfect regularity. For a classical coin with real bias p, the probability of getting exactly k heads and z tails in n throws is

$$\binom{n}{k} p^k \, (1-p)^{n-k},$$

which generally cannot be 1 unless p is either 0 or 1 (and $k \in \{0, n\}$).

Polynomial Argument. Setting

$$\binom{n}{k} p^k (1-p)^{n-k} \;-\; 1 \;=\; 0$$

yields a polynomial equation of degree n in p. Typically, such a polynomial has up to n complex solutions, and real solutions are either trivial or do not exist for nontrivial (k, z). Hence, to exactly match this urn model's behavior by a single "coin," one must allow that coin to have a *complex* bias p. In other words, a classical (real-valued) coin cannot replicate the phenomenon where *every* block of n trials yields exactly k heads and z tails with probability 1.

Reversing the Perspective

One may ask: for every complex root p of the above polynomial, does there exist a concrete urn process (or another classical but *dependent* Bernoulli scheme) that reproduces the effect of a coin with complex probability p?

To clarify this, consider the case $n = 2$ and $k = 1$. The relevant polynomial is

$$2\,p(1-p) \;-\; 1 \;=\; 0 \quad\Longleftrightarrow\quad p^2 \;-\; p \;+\; \tfrac{1}{2} \;=\; 0.$$

Its solutions are

$$p_{1,2} \;=\; \tfrac{1}{2} \;\pm\; \tfrac{i}{2}.$$

Hence, if a "coin" had $p = \frac{1}{2} + \frac{i}{2}$ as its probability of landing Heads (and $q = 1 - p$ for Tails), we would want the two successive outcomes (first draw, second draw) to be complementary with probability 1. In classical terms, that means $P(\text{opposite outcomes}) = 1$. One can realize this via a pair of *dependent* coins such that:

$$P(X_1 = 0,\, X_2 = 1) \;=\; P(X_1 = 1,\, X_2 = 0) \;=\; \tfrac{1}{2},$$

and

$$P(X_1 = 0,\, X_2 = 0) \;=\; 0, \quad P(X_1 = 1,\, X_2 = 1) \;=\; 0.$$

These two coins obviously cannot be independent (by standard product-rule contradictions). Instead, they form a simple 2-state Markov chain that always flips from one state to the other. This classical process effectively "simulates" a single coin with a complex bias of $p = \frac{1}{2} + \frac{i}{2}$.

Connection to the Urn. In the urn scenario with $k = 1$ and $z = 1$, we have exactly 2 balls in total. The procedure ensures that after the first draw (Heads or Tails), the second draw is guaranteed to be the other type. This yields an identical pattern of outcomes. Thus, the urn model aligns with the idea of a "complex probability" for heads in the sense that it enforces a perfect 2-draw complementarity.

2 Conclusion

We have demonstrated that an urn model with $(k + z)$ draws per cycle can exhibit 100% likelihood of obtaining exactly k Heads-balls and z Tails-balls in each block of $n = k + z$. This phenomenon cannot be replicated by any standard coin with a fixed real bias p. Instead, one must allow for complex solutions to the binomial equation

$$\binom{n}{k}\, p^k (1-p)^{n-k} \;-\; 1 \;=\; 0,$$

leading to the concept of a "coin with complex probability."

Moreover, we have seen that such a complex-biased coin can be mirrored by a pair (or more generally, an n-tuple) of interdependent classical random variables. In the simplest case $(n = 2, k = 1)$, the coin's imaginary bias emerges from a classical Markov chain that forces complementary outcomes. Hence, complex probabilities can sometimes be interpreted purely in terms of classical but correlated processes, bridging the gap between real probability theory and formal complex extensions.

3 Philosophical Implications

- **Extended Notion of Probability.** Allowing complex-valued probabilities challenges the usual Kolmogorov framework, wherein probabilities lie in $[0, 1]$. Philosophically, one might question whether these complex values have any direct physical meaning or whether they serve merely as a formal tool for describing correlations that cannot be represented by real-valued biases.

- **Dependence vs. Complex Amplitudes.** The urn model illustrates that phenomena suggestive of "complex probability" can be realized by introducing *dependence* between draws. This resonates with certain quantum-mechanical analogies, where complex amplitudes reflect entanglement or correlations that defy a naive product structure.

- **Interpretation in Probability Theory.** From a foundational standpoint, the idea that one can encode complex probabilities in pairs of dependent Bernoulli trials questions how strictly we should interpret probabilities as frequencies. Do imaginary parts correspond to some hidden or relational property of the system that only manifests via correlation?
- **Analogy to Construction of $\mathbb{C}$.** Just as complex numbers can be realized as ordered pairs of real numbers with special algebraic rules, so too can "complex probabilities" be realized by paired or n-tuple processes in classical probability. This parallel raises the deeper question of whether every formal extension of probability might be given a classical realization by suitably complicated dependencies.